Innermost Reachability and Context Sensitive Reachability Properties are Decidable for Linear Right-Shallow Term Rewriting Systems

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Abstract. A reachability problem is a problem used to decide whether s is reachable to t by R or not for a given two terms s, t and a term rewriting system R. Since it is known that this problem is undecidable, effort has been devoted to finding subclasses of term rewriting systems in which the reachability is decidable. However few works on decidability exist for innermost reduction strategy or context-sensitive rewriting.
In this paper, we show that innermost reachability and context-sensitive reachability are decidable for linear right-shallow term rewriting systems. Our approach is based on the tree automata technique that is commonly used for analysis of reachability and its related properties.

1 Introduction

The reachability problem is a problem used to decide whether s is reachable to t by R or not for a given two terms: s, t, and a term rewriting systems (TRS) R. Since it is known that this problem is undecidable even if restricted to linear TRS or to shallow TRS [7], effort has been made to find subclasses of TRSs in which the reachability is decidable. Reachability properties for several subclasses of TRSs have been proved to be decidable [2, 6, 10, 11, 3]. These results are based on the more powerful property of effective preservation of regularity. We say a rewrite relation effectively preserves regularity if it is possible to construct a tree automaton (TA) which recognizes a set of terms reachable from some term in the regular set defined by a given TA. It is easy to see that the reachability property for TRSs is decidable if the TRSs effectively preserve regularity.

Innermost reduction, a strategy that rewrites innermost redexes, is used for call-by-value computation. Context-sensitive reduction [8] is a strategy in which rewritable positions are indicated by specifying arguments of function symbols. For innermost reduction strategy, recently Godoy and Huntingford showed that reachability and joinability with respect to innermost reduction for (possibly non-linear) shallow TRSs are decidable [5]. In this case the proof method is not based on tree automata techniques. This paper shows that innermost reduction and context-sensitive reduction effectively preserve regularity for linear right-shallow term rewriting systems: hence, innermost reachability, innermost joinability, context-sensitive joinability and context-sensitive reachability are decidable for this class.

2 Preliminary

Let F be a set of function symbols with fixed arity and X be an enumerable set of variables. The arity of function symbol f is denoted by $\mathrm{ar}(f)$. Function symbols with $\mathrm{ar}(f) = 0$ are *constants*. The set of *terms*, defined in the usual way, is denoted by $\mathcal{T}(F, X)$. A term is *linear* if no variable occurs more than once in the term. The set of variables occurring in t is denoted by $\mathrm{Var}(t)$. A term t is *ground* if $\mathrm{Var}(t) = \emptyset$. The set of ground terms is denoted by $\mathcal{T}(F)$.

A *position* in a term t is defined, as usual, as a sequence of positive integers, and the set of all positions in a term t is denoted by $\mathrm{Pos}(t)$, where the empty sequence ε is used to denote root position. The depth of a position p is denoted by $|p|$. A term t is *shallow* if every variable occurs at depth 0 or 1 in t. The *subterm* of t at position p is denoted by $t|_p$, and $t[t']_p$ represents the term obtained from t by replacing the subterm $t|_p$ by t'.

A *substitution* σ is a mapping from X to $\mathcal{T}(F, X)$ whose domain $\mathrm{Dom}(\sigma) = \{x \in X \mid x \neq \sigma(x)\}$ is finite. We sometimes represent σ as $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ where $x_i \in \mathrm{Dom}(\sigma)$ and $t_i = \sigma(x_i)$. The term obtained by applying a substitution σ to a term t is written as $t\sigma$.

A *rewrite rule* is an ordered pair of terms in $\mathcal{T}(F, X)$, written as $l \to r$, where $l \notin X$ and $\mathrm{Var}(l) \supseteq \mathrm{Var}(r)$. We say that variables $x \in \mathrm{Var}(l) \setminus \mathrm{Var}(r)$ are *erasing*. A *term rewriting system (over F)* (TRS) is a finite set of rewrite rules. Rewrite relation $\underset{R}{\to}$ induced by a TRS R is as follows: $s \underset{R}{\to} t$ if and only if $s = s[l\sigma]_p$, and $t = s[r\sigma]_p$ for some rule $l \to r \in R$, with substitution σ and position $p \in \mathrm{Pos}(s)$. We call $l\sigma$ *redex*. We sometimes write $\underset{R}{\to}^p$ by presenting the position p explicitly.

A rewrite rule $l \to r$ is *left-linear* (resp. *right-linear*, *linear*, *right-shallow*) if l is linear (resp. r is linear, l and r are linear, r is shallow). A TRS R is *left-linear* (resp. *right-linear*, *linear*, *right-shallow*) if every rule in R is left-linear (resp. right-linear, linear, right-shallow).

A *tree automaton* (TA) is a 4-tuple $\mathcal{A} = (F, Q, Q^f, \Delta)$ where Q is a finite set of states, $Q^f (\subseteq Q)$ is a set of final states, and Δ is a finite set of transition rules of the forms $f(q_1, \ldots, q_n) \to q$ or $q_1 \to q$ where $f \in F$ with $\mathrm{ar}(f) = n$, and $q_1, \ldots, q_n, q \in Q$. We can regard Δ as a TRS over $F \cup Q$. The rewrite relation induced by Δ is called a *transition relation* denoted by $\underset{\Delta}{\to}$ or $\underset{\mathcal{A}}{\to}$. We say that a term s $(\in \mathcal{T}(F))$ is *accepted* by $\mathcal{A}$ if $s \overset{*}{\underset{\mathcal{A}}{\to}} q \in Q^f$. The set of all terms accepted by $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$. We say $\mathcal{A}$ *recognizes* $\mathcal{L}(\mathcal{A})$. We use a notation $\mathcal{L}(\mathcal{A}, q)$ or $\mathcal{L}(\Delta, q)$ to represent the set $\{s \mid s \overset{*}{\underset{\mathcal{A}}{\to}} q\}$. A TA $\mathcal{A}$ is *deterministic* if $s \overset{*}{\underset{\mathcal{A}}{\to}} q$ and $s \overset{*}{\underset{\mathcal{A}}{\to}} q'$ implies $q = q'$ for any $s \in \mathcal{T}(F)$. A TA $\mathcal{A}$ is *complete* if there exists $q \in Q$ such that $s \overset{*}{\underset{\mathcal{A}}{\to}} q$ for any $s \in \mathcal{T}(F)$. A set T of terms is *regular* if there exists a TA $\mathcal{A}$ such that $T = \mathcal{L}(\mathcal{A})$.

Let $\to$ be a binary relation on a set $\mathcal{T}(F)$. We say $s \in \mathcal{T}(F)$ is a *normal form* (with respect to $\to$) if there exists no term $t \in \mathcal{T}(F)$ such that $s \to t$. We use $\circ$ to denote the composition of two relations. We write $\xrightarrow{*}$ for the reflexive and transitive closure of $\to$. We also write $\xrightarrow{n}$ for the relation $\to \circ \cdots \circ \to$ that is composed of n $\to$'s. The set of *reachable terms* from a term in T is defined by $\to[T] = \{t \mid s \in T,\ s \xrightarrow{*} t\}$. We say that a reduction $\to$ *effectively preserves regularity* if a tree automata $\mathcal{A}_*$ that satisfies $\mathcal{L}(\mathcal{A}_*) = \to[\mathcal{L}(\mathcal{A})]$ can be effectively constructed from an automata $\mathcal{A}$. The *reachability problem* (resp. *joinability problem*) with respect to $\to$ is a problem that decides whether $s \xrightarrow{*} s'$ (resp. $s \xrightarrow{*} \circ \xleftarrow{*} s'$) or not, for given terms s and s'.

Theorem 1 ([4]). *Let $\to$ be a relation on terms that effectively preserves recognizability. Then both reachability and joinability properties with respect to $\to$ are decidable.*

3 Regularity preservation for innermost reduction

We say a step rewrite $s \xrightarrow[R]{}^{p} t$ is *innermost* if all proper subterms of $s|_p$ are normal forms. We write $\xrightarrow[R]{}_{\text{in}}$ for the innermost rewrite relation induced by R.

This section shows that innermost reduction $\xrightarrow[R]{}_{\text{in}}$ effectively preserves regularity if R is a linear right-shallow TRS. In order to show the property, we prepare a procedure $\mathrm{P_{in}}$ that inputs a TA $\mathcal{A}$ and a TRS R and outputs a TA $\mathcal{A}_*$, and show that $\mathcal{A}_*$ recognizes a set $\xrightarrow[R]{}_{\text{in}}[\mathcal{L}(\mathcal{A})]$. The procedure almost follows the procedure in [6]. The main difference is the construction of states. Each state in the resulting automata consists of a pair of states. The first state originates in the input automata and remembers a reachable set. The second state remembers whether the corresponding terms are a normal form or not, which is necessary because every proper subterm of the innermost redex must be a normal form. First we show an example.

Example 2. Let $R = \{a \to b,\ f(x) \to g(x)\}$ and $\mathcal{A}$ be a TA such that $\mathcal{L}(\mathcal{A}) = \{f(a)\}$ defined by a finite state $\{q_{fa}\}$, and transition rules $\{a \to q_a,\ f(q_a) \to q_{fa}\}$. The procedure produces the following TA defined by final states: $Q^f_* = \{\langle q_{fa}, u_a\rangle,\ \langle q_{fa}, u_b\rangle\}$ and transition rules: $\Delta_* = \{a \to \langle q_a, u_a\rangle, b \to \langle q_a, u_b\rangle, f(\langle q_a, u_a\rangle) \to \langle q_{fa}, u_a\rangle, f(\langle q_a, u_b\rangle) \to \langle q_{fa}, u_a\rangle, g(\langle q_a, u_b\rangle) \to \langle q_{fa}, u_b\rangle\}$. Here u_b is a state for normal forms.

The TA $\mathcal{A}_*$ accepts terms $f(a)$, $f(b)$ and $g(b)$ in $\xrightarrow[R]{}_{\text{in}}[\{f(a)\}]$, and does not accept $g(a)$. □

We show the procedure $\mathrm{P_{in}}$ in Figure 1, where we use a notation $\mathrm{RS}(R)$ for the set of all *non-variable direct subterms of the right-hand sides* of rules in TRS R: that is, $\mathrm{RS}(R) = \{r_i \notin X \mid l \to f(r_1, \ldots, r_n) \in R\}$. Note that $\mathrm{RS}(R)$ is a set of ground terms if R is right shallow.

Example 3. Let us follow how procedure $\mathrm{P_{in}}$ works. Consider R and $\mathcal{A}$ in Example 2.

Input TA $\mathcal{A} = \langle F, Q, Q^f, \Delta \rangle$ and left-linear right-shallow TRS R over F.
Output TA $\mathcal{A}_* = \langle F, Q_*, Q_*^f, \Delta_* \rangle$ such that $\mathcal{L}(\mathcal{A}_*) = \xrightarrow[R]{}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$ if R is right-linear.

Step 1 (initialize) 1. Prepare a TA $\mathcal{A}_{\mathrm{RS}} = \langle F, Q_{\mathrm{RS}}, Q_{\mathrm{RS}}^f, \Delta_{\mathrm{RS}} \rangle$ such that $Q_{\mathrm{RS}}^f = \{q^t \mid t \in \mathrm{RS}(R)\}$ and $\mathcal{L}(\mathcal{A}_{\mathrm{RS}}, q^t) = \{t\}$.

2. Prepare a deterministic complete TA $\mathcal{A}_{\mathrm{NF}} = \langle F, Q_{\mathrm{NF}}, Q_{\mathrm{NF}}^f, \Delta_{\mathrm{NF}} \rangle$ such that
   - $\mathcal{L}(\mathcal{A}_{\mathrm{NF}})$ is the set NF_R ($\subseteq \mathcal{T}(F)$) of all ground normal forms
   - $\mathcal{L}(\mathcal{A}_{\mathrm{NF}}, q) \neq \emptyset$ for any $q \in Q_{\mathrm{NF}}$.
3. Let
   - $k := 0$
   - $Q_* = (Q \uplus Q_{\mathrm{RS}}) \times Q_{\mathrm{NF}}$
   - $Q_*^f = Q^f \times Q_{\mathrm{NF}}$
   - $\Delta_0 = \{ f(\langle q_1, u_1 \rangle, \ldots, \langle q_n, u_n \rangle) \to \langle q, u \rangle \mid$
     $f(q_1, \ldots, q_n) \to q \in \Delta \uplus \Delta_{\mathrm{RS}},\ f(u_1, \ldots, u_n) \to u \in \Delta_{\mathrm{NF}} \}$

Step 2 Let Δ_{k+1} be transition rules produced by augmenting transition rules of Δ_k by the following inference rules:

$$\frac{f(l_1, ..., l_n) \to g(r_1, ..., r_m) \in R,\ f(\langle q_1, u_1 \rangle, \ldots, \langle q_n, u_n \rangle) \to \langle q, u \rangle \in \Delta_k}{g(\langle q_1', u_1' \rangle, \ldots, \langle q_m', u_m' \rangle) \to \langle q, u' \rangle \in \Delta_{k+1}}$$

if there exists $\theta : X \to (Q \uplus Q_{\mathrm{RS}}) \times Q_{\mathrm{NF}}^f$ such that

- $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, u_i \rangle$ and $u_i \in Q_{\mathrm{NF}}^f$ for all $1 \leq i \leq n$,
- $\langle q_j', u_j' \rangle = \begin{cases} r_j\theta & \cdots\ r_j \in X \\ \langle q^{r_j}, u'' \rangle & \cdots\ r_j \notin X,\ u'' \in Q_{\mathrm{NF}} \end{cases}$ for all $1 \leq j \leq m$, and
- $g(u_1', \ldots, u_m') \xrightarrow[\Delta_{\mathrm{NF}}]{} u'$.

and

$$\frac{f(l_1, ..., l_n) \to x \in R,\ f(\langle q_1, u_1 \rangle, \ldots, \langle q_n, u_n \rangle) \to \langle q, u \rangle \in \Delta_k}{\langle q', u' \rangle \to \langle q, u' \rangle \in \Delta_{k+1}}$$

if there exists $\theta : X \to (Q \uplus Q_{\mathrm{RS}}) \times Q_{\mathrm{NF}}^f$ such that

- $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, u_i \rangle$ for all $1 \leq i \leq n$, and
- $\langle q', u' \rangle = x\theta$.

Step 3 If $\Delta_{k+1} = \Delta_k$ then stop and set $\Delta_* = \Delta_k$. Otherwise, $k := k + 1$, and go to step 2.

Fig. 1. Procedure $\mathrm{P_{in}}$

In the initialization step, we have $\Delta_{\mathrm{RS}} = \emptyset$, $Q^f_{\mathrm{NF}} = \{u_b\}, \Delta_{\mathrm{NF}} = \{a \to u_a, b \to u_b, f(u_a) \to u_a, f(u_b) \to u_a, g(u_a) \to u_a, g(u_b) \to u_b\}$, $Q_* = \{q_a, q_{fa}\} \times \{u_a, u_b\}$, $Q^f_* = \{q_{fa}\} \times \{u_a, u_b\}$, $\Delta_0 = \{a \to \langle q_a, u_a\rangle, f(\langle q_a, u_a\rangle) \to \langle q_{fa}, u_a\rangle, f(\langle q_a, u_b\rangle) \to \langle q_{fa}, u_a\rangle\}$.

The saturation steps stop at $k = 1$ and we have $\Delta_1 = \Delta_0 \cup \{b \to \langle q_a, u_b\rangle,\ g(\langle q_a, u_b\rangle) \to \langle q_{fa}, u_b\rangle\}$, $\Delta_2 = \Delta_1$ □

The procedure $\mathrm{P_{in}}$ eventually terminates at some k, because rewrite rules in R and states Q_* are finite, and hence, possible transitions rules are finite. Apparently $\Delta_0 \subset \cdots \subset \Delta_k = \Delta_{k+1} = \cdots$. A measurement of transitions of Δ_* is defined as $||s \xrightarrow[\Delta_0]{} t|| = 0$ and $||s \xrightarrow[\Delta_{i+1}\setminus\Delta_i]{} t|| = i+1$ for $i \geq 0$. This is extended on transition sequences as a multiset:

$$||s_0 \xrightarrow[\Delta_*]{} s_1 \xrightarrow[\Delta_*]{} \cdots \xrightarrow[\Delta_*]{} s_n|| = \{||s_i \xrightarrow[\Delta_*]{} s_{i+1}|| \mid 0 \leq i < n\}.$$

Now we can define an order $\sqsupset$ on transition sequences by Δ_*, which is necessary in proofs.

$$\alpha \sqsupset \beta \overset{\mathrm{def}}{\Leftrightarrow} ||\alpha|| >_{\mathrm{mul}} ||\beta||$$

where $>_{\mathrm{mul}}$ is the multiset extension of $>$ on $\mathbb{N}$.

Proposition 4. *(a)* $s \xrightarrow[\Delta]{*} q$ *if and only if* $s \xrightarrow[\Delta_0]{*} \langle q, u\rangle$ *for some* $u \in Q_{NF}$.
(b) $s \xrightarrow[\Delta_*]{*} \langle q, u\rangle$ *implies* $s \xrightarrow[\Delta_{\mathrm{NF}}]{*} u$

Proof. Direct consequence of the construction of Δ_0 and the completeness of $\mathcal{A}_{\mathrm{NF}}$. □

Lemma 5. *Let* $\alpha:\ s[\langle q, u\rangle]_p \xrightarrow[\Delta_k]{*} \langle q', u'\rangle$. *If* $k = 0$ *or* $u \in Q_{NF} \setminus Q^f_{NF}$ *then there exists* $v' \in Q_{NF}$ *such that* $\beta:\ s[\langle q, v\rangle]_p \xrightarrow[\Delta_k]{*} \langle q', v'\rangle$ *and* $\alpha \sqsupseteq \beta$ *for any* $v \in Q_{NF}$.

Proof. If $k = 0$, it trivially holds from the construction of Δ_0. We prove in the case $u \in Q_{\mathrm{NF}} \setminus Q^f_{\mathrm{NF}}$ by induction on steps n of transition $s[\langle q, v\rangle]_p \xrightarrow[\Delta_k]{n} \langle q', v'\rangle$. Since it is trivial in the case $n = 0$, let $n > 0$. Then we have two cases according to the form of the transition rule applied in the last step.

1. In the case that

$$s[\langle q, u\rangle]_p \xrightarrow[\Delta_k]{n-1} f(\langle q_1, u_1\rangle, \ldots, \langle q_n, u_n\rangle) \xrightarrow[\Delta_k]{} \langle q', u'\rangle, \tag{1}$$

   the position p can be represented as ip' for $1 \leq i \leq n$. Here $u_i \notin Q^f_{\mathrm{NF}}$ follows from $u \notin Q^f_{\mathrm{NF}}$, Proposition 4 (b), and the construction of Δ_{NF}. Since $\alpha_i : (s|_i)[\langle q, u\rangle]_{p'} \xrightarrow[\Delta_k]{*} \langle q_i, u_i\rangle$, we have $\beta_i : (s|_i)[\langle q, v\rangle]_{p'} \xrightarrow[\Delta_k]{*} \langle q_i, v''\rangle$ and $\alpha_i \sqsupseteq \beta_i$ for some $v'' \in Q_{\mathrm{NF}}$ by the induction hypothesis. Thus we have $s[\langle q, v\rangle]_p \xrightarrow[\Delta_k]{*} f(\ldots, \langle q_{i-1}, u_{i-1}\rangle, \langle q_i, v''\rangle, \langle q_{i+1}, u_{i+1}\rangle, \ldots)$
   (a) If the transition rule in the last step of (1) is in Δ_0, we also have $f(\ldots, \langle q_{i-1}, u_{i-1}\rangle, \langle q_i, v''\rangle, \langle q_{i+1}, u_{i+1}\rangle, \ldots) \to \langle q', v'\rangle \in \Delta_0$ from the construction, where v' is determined by $f(\ldots, u_{i-1}, v'', u_{i+1}, \ldots) \to v' \in \Delta_{\mathrm{NF}}$.

(b) Otherwise we assume that the transition rule in the last step of (1) is in $\Delta_k \setminus \Delta_{k-1}$ without loss of generality. It is known that the rule is produced by the first inference rule in Step 2 and hence $r_i \notin X$; otherwise $u_i \in Q^f_{\mathrm{NF}}$ follows from $\langle q_i, u_i\rangle = r_i\theta$, which contradicts $u_i \notin Q^f_{\mathrm{NF}}$. Thus $f(\dots, \langle q_{i-1}, u_{i-1}\rangle, \langle q_i, v''\rangle, \langle q_{i+1}, u_{i+1}\rangle, \dots) \to \langle q', v'\rangle \in \Delta_k \setminus \Delta_{k-1}$ and $\alpha \sqsupseteq \beta$.

2. In the case that $s[\langle q, u\rangle]_p \xrightarrow[\Delta_k]{n-1} \langle q_1, u_1\rangle \xrightarrow[\Delta_k]{} \langle q', u'\rangle$, we can show it similarly to the previous case. □

Lemma 6. *Let R be left-linear and right-shallow. Then $s \xrightarrow[\Delta_*]{*} \langle q, u\rangle$ and $s \xrightarrow[R]{n}_{\mathrm{in}} t$ imply $t \xrightarrow[\Delta_*]{*} \langle q, u'\rangle$ for some $u' \in Q_{NF}$.*

Proof. We present the proof in the case where $n = 1$. Let $s \xrightarrow[\Delta_k]{*} \langle q, u\rangle$ and $s = s[l\sigma]_p \xrightarrow[R]{}_{\mathrm{in}} s[r\sigma]_p = t$ for some rewrite rule $l \to r \in R$.

1. Consider the case where the rewrite rule is in the form $f(l_1, \dots, l_n) \to g(r_1, \dots, r_m)$. Since this rewrite rule is left-linear, $s \xrightarrow[\Delta_k]{*} \langle q, u\rangle$ is represented as $s = s[f(l_1\sigma, \dots, l_n\sigma)]_p \xrightarrow[\Delta_k]{*} s[f(l_1\theta, \dots, l_n\theta)]_p \xrightarrow[\Delta_k]{*} s[f(\langle q_1, u_1\rangle, \dots, \langle q_n, u_n\rangle)]_p \xrightarrow[\Delta_k]{} s[\langle q'', u''\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, u\rangle$ for some $\theta : X \to Q_*$. Note that $u'' \notin Q^f_{\mathrm{NF}}$ since $s|_p$ is not a normal form.
We have $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, u_i\rangle$, where $u_i \in Q^f_{\mathrm{NF}}$ since each $l_i\sigma$ is a normal form. We also have $f(u_1, \dots, u_n) \xrightarrow[\Delta_{\mathrm{NF}}]{*} u''$ by Proposition 4 (b). From the construction of $\mathcal{A}_*$, there exists a transition rule $g(\langle q'_1, u'_1\rangle, \dots, \langle q'_n, u'_m\rangle) \to \langle q'', v'\rangle \in \Delta_{k+1}$ such that
$$\langle q'_j, u'_j\rangle = \begin{cases} r_j\theta & \cdots\ r_j \in X \\ \langle q^{r_j}, v''\rangle & \cdots\ r_j \notin X \text{ where } r_j \xrightarrow[\Delta_0]{*} \langle q^{r_j}, v''\rangle \end{cases}$$
   (a) For j such that $r_j \in X$, we have $l_i|_{p'} = r_j$ for some i and p'. Hence $r_j\sigma = l_i|_{p'}\sigma \xrightarrow[\Delta_k]{*} l_i|_{p'}\theta = r_j\theta = \langle q'_j, u'_j\rangle$.
   (b) For j such that $r_j \notin X$, we have $r_j\sigma = r_j$ since R is right-shallow, and $r_j \xrightarrow[\Delta_0]{*} \langle q^{r_j}, v''\rangle = \langle q'_j, u'_j\rangle$.
   Therefore we have $t = s[g(r_1\sigma, \dots, r_m\sigma)]_p \xrightarrow[\Delta_k]{*} s[g(\langle q'_1, u'_1\rangle, \dots, \langle q'_m, u'_m\rangle)]_p \xrightarrow[\Delta_{k+1}]{} s[\langle q'', v'\rangle]_p$, and $s[\langle q'', v'\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, u'\rangle$ for some $u' \in Q_{\mathrm{NF}}$ by Lemma 5.
2. In the case that the rewrite rule is in the form $f(l_1, \dots, l_n) \to x$, we can show it similarly to the previous case. □

Now we obtain the following lemma.

Lemma 7. *If R be left-linear and right-shallow, then $\mathcal{L}(\mathcal{A}_*) \supseteq \xrightarrow[R]{}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$.*

Proof. Let $s \xrightarrow[R]{*}_{\mathrm{in}} t$ and $s \xrightarrow[\Delta]{*} q \in Q^f$. Then we have $s \xrightarrow[\Delta_0]{*} \langle q, u\rangle \in Q^f_*$ by Proposition 4 (a). Hence $t \xrightarrow[\Delta_*]{*} \langle q, u'\rangle \in Q^f_*$ by Lemma 6. □

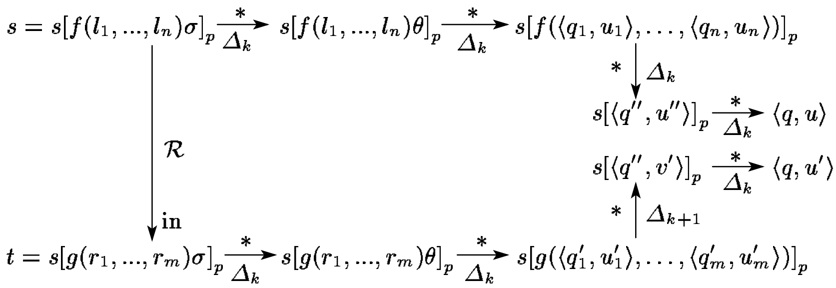

Fig. 2. The diagram of proof of lemma 6

Lemma 8. *Let* Δ_* *be generated from a right-linear right-shallow TRS. Then* $\alpha: t\ (\xrightarrow[\Delta_0]{*} \circ \xrightarrow[\Delta_{k+1}]{})\ \langle q, u'\rangle$ *implies* $s \xrightarrow[R]{}_{\text{in}} t$, $\beta: s \xrightarrow[\Delta_k]{*} \langle q, u\rangle$ *and* $\alpha \sqsupset \beta$ *for some term* s *and* $u \in Q_{NF}$.

Proof. Consider the case where the last transition rule applied in α is (in the form of) $g(\langle q'_1, u'_1\rangle, \ldots, \langle q'_m, u'_m\rangle) \rightarrow \langle q, u'\rangle$ and we assume that it is in $\Delta_{k+1} \setminus \Delta_k$ without loss of generality. Then α can be represented as $t = g(t_1, \ldots, t_m) \xrightarrow[\Delta_0]{*} g(\langle q'_1, u'_1\rangle, \ldots, \langle q'_m, u'_m\rangle) \xrightarrow[\Delta_{k+1}\setminus\Delta_k]{} \langle q, u'\rangle$, and the last transition rule applied is added by the first inference rule in the procedure. Hence there exist $f(l_1, \ldots, l_n) \rightarrow g(r_1, \ldots, r_m) \in R$, $f(\langle q_1, u_1\rangle, \ldots, \langle q_n, u_n\rangle) \rightarrow \langle q, u\rangle \in \Delta_k$, and $\theta: X \rightarrow (Q \uplus Q_{\text{RS}}) \times Q^f_{\text{NF}}$ such that

- $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, u_i\rangle$ and $u_i \in Q^f_{\text{NF}}$,
- $\langle q'_j, u'_j\rangle = r_j\theta$ if $r_j \in X$,
- $q'_j = q^{r_j}$ if $r_j \notin X$, and
- $\mathcal{L}(\Delta_0, x\theta) \neq \emptyset$ for each erasing variable.

Hence, we have the following:

1. For j such that $r_j \in X$, we have $t_j \xrightarrow[\Delta_0]{*} \langle q'_j, u'_j\rangle = r_j\theta$.
2. For j such that $r_j \notin X$, we have $t_j \xrightarrow[\Delta_{\text{RS}}]{*} q^{r_j}$ hence $t_j = r_j$ from the construction of Δ_0. Thus we have $t_j = r_j\theta$ since R is right-shallow.

Thus we have $g(t_1, \ldots, t_m) \xrightarrow[\Delta_0]{*} g(r_1\theta, \ldots, r_m\theta) \xrightarrow[\Delta_0]{*} g(\langle q'_1, u'_1\rangle, \ldots, \langle q'_m, u'_m\rangle)$. We define a substitution $\sigma:\ \text{Var}(f(l_1, \ldots, l_n)) \rightarrow \mathcal{T}(F)$ as follows:

$$x\sigma = \begin{cases} t_j & \cdots \text{ if there exists } j \text{ such that } r_j = x \\ t' & \cdots \text{ otherwise, choose an arbitral } t' \text{ such that } t' \xrightarrow[\Delta_0]{*} x\theta, \end{cases}$$

where σ is well-defined from the right-linearity of rewrite rules. We can construct $\beta: f(l_1, \ldots, l_n)\sigma \xrightarrow[\Delta_0]{*} f(l_1, \ldots, l_n)\theta \xrightarrow[\Delta_k]{*} f(\langle q_1, u_1\rangle, \ldots, \langle q_n, u_n\rangle) \xrightarrow[\Delta_k]{} \langle q, u\rangle$, where $\alpha \sqsupset \beta$. Since $u_i \in Q^f_{\text{NF}}$, each $l_i\sigma$ is a normal form. Hence we have

$f(l_1,\ldots,l_n)\sigma \xrightarrow[R]{}_{\text{in}} g(r_1,\ldots,r_m)\sigma = g(t_1,\ldots,t_m) = t$. Therefore the lemma of the case follows by taking $s = f(l_1,\ldots,l_n)\sigma$.

For the case where the transition rule applied last in α is (in the form of) $\langle q', u''\rangle \to \langle q, u'\rangle$, the lemma can be shown as similar to the previous case. $\square$

Lemma 9. *If R be right-linear and right-shallow, then $\mathcal{L}(\mathcal{A}_*) \subseteq \xrightarrow[R]{}_{\text{in}}[\mathcal{L}(\mathcal{A})]$.*

Proof. From Proposition 4(a), it is enough to show the claim that $\alpha : t \xrightarrow[\Delta_*]{*} \langle q, u'\rangle$ implies $s \xrightarrow[R]{*}_{\text{in}} t$, and $s \xrightarrow[\Delta_0]{*} \langle q, u\rangle$ for some $s \in \mathcal{T}(F)$ and $u \in Q_{\text{NF}}$. We prove it by induction on α with respect to $\sqsupset$.

1. Consider the case where the last transition rule applied in α is (in the form of) $g(\langle q'_1, u'_1\rangle,\ldots,\langle q'_m, u'_m\rangle) \to \langle q, u\rangle \in \Delta_k$. Then α can be represented as $t = g(t_1,\ldots,t_m) \xrightarrow[\Delta_*]{*} g(\langle q'_1, u'_1\rangle,\ldots,\langle q'_m, u'_m\rangle) \xrightarrow[\Delta_k]{} \langle q, u\rangle$. Since $\alpha \sqsupset (t_j \xrightarrow[\Delta_*]{*} \langle q'_j, u'_j\rangle)$, there exists s_j for every j such that $s_j \xrightarrow[R]{*}_{\text{in}} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle q'_j, v'_j\rangle$ from the induction hypothesis. Here we have $v'_j \in Q_{\text{NF}} \setminus Q^f_{\text{NF}}$ or $s_j = t_j$ for each j since $v'_i \in Q^f_{\text{NF}}$ implies that s_i is a normal form. Hence $g(s_1,\ldots,s_m) \xrightarrow[R]{*}_{\text{in}} t$ and we have $\alpha' : g(s_1,\ldots,s_m) \xrightarrow[\Delta_0]{*} g(\langle q'_1, v'_1\rangle,\ldots,\langle q'_m, v'_m\rangle) \xrightarrow[\Delta_k]{} \langle q, v\rangle$ and $\alpha \sqsupseteq \alpha'$ by applying Lemma 5 repeatedly to $g(\langle q'_1, u'_1\rangle,\ldots,\langle q'_m, u'_m\rangle) \xrightarrow[\Delta_k]{} \langle q, u\rangle$.
   If $k = 0$ then the claim trivially holds by letting $s = g(s_1,\ldots,s_m)$. Hence let $k > 0$. Then we have $\beta : s' \xrightarrow[\Delta_{k-1}]{*} \langle q, v'\rangle$ with $\alpha' \sqsupset \beta$ and $s' \xrightarrow[R]{}_{\text{in}} g(s_1,\ldots,s_m)$ for some s' and v' by Lemma 8. Since $\alpha \sqsupset \beta$, the claim of this case follows from the induction hypothesis.
2. In the case where the last transition rule applied in α is (in the form of) $\langle q', u'\rangle \to \langle q, u\rangle \in \Delta_k$, we can show it similarly to the previous case. $\square$

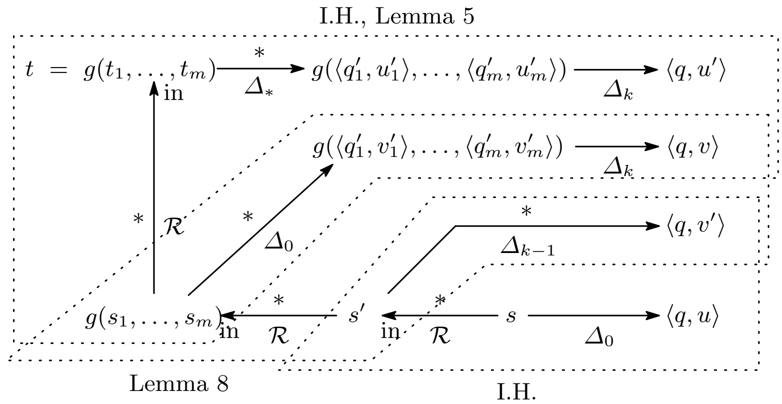

Fig. 3. The diagram of proof of lemma 9

We obtain the following theorems from Lemma 7, Lemma 9, and Theorem 1.

Theorem 10. *Innermost reduction for linear right-shallow TRSs effectively preserves regularity. Thus innermost reachability is a decidable property for linear right-shallow TRSs.*

4 Regularity preservation for context-sensitive reduction

A context-sensitive rewrite relation is a subrelation of a rewrite relation in which rewritable positions are indicated by specifying arguments of function symbols. A mapping $\mu : F \to \mathcal{P}(\mathbb{N})$ is said to be a *replacement map* (or F-map) if $\mu(f) \subseteq \{1, \ldots, \mathrm{ar}(f)\}$ for all $f \in F$. A *context-sensitive rewriting system* (CS-TRS) is a pair $\mathcal{R} = (R, \mu)$ of a TRS and a replacement map. We say R is an underlined TRS of $\mathcal{R}$. The set of *μ-replacing positions* $\mathrm{Pos}^{\mu}(t)$ ($\subseteq \mathrm{Pos}(t)$) is recursively defined: $\mathrm{Pos}^{\mu}(t) = \{\varepsilon\}$ if t is a constant or a variable, otherwise $\mathrm{Pos}^{\mu}(f(t_1, \ldots, t_n)) = \{\varepsilon\} \cup \{ip \mid i \in \mu(f), p \in \mathrm{Pos}^{\mu}(t_i)\}$. The rewrite relation induced by a CS-TRS $\mathcal{R}$ is defined: $s \underset{\mathcal{R}}{\hookrightarrow} t$ if and only if $s \xrightarrow[R]{}^{p} t$ and $p \in \mathrm{Pos}^{\mu}(t)$.

Similarly to the previous section, this section shows that a context-sensitive reduction $\underset{\mathcal{R}}{\hookrightarrow}$ effectively preserves recognizability if the underlined TRS R of $\mathcal{R}$ is linear and right-shallow. In order to show the property, we prepare a procedure $\mathrm{P_{cs}}$ that inputs a TA $\mathcal{A}$ and a CS-TRS $\mathcal{R}$ and outputs a TA $\mathcal{A}_*$, and show that $\mathcal{L}(\mathcal{A}_*) = \underset{\mathcal{R}}{\hookrightarrow}[\mathcal{L}(\mathcal{A})]$.

The main idea is the introduction of an extra state $\tilde{q}$ for each state q. The former state $\tilde{q}$ is used for accepting terms in $\underset{\mathcal{R}}{\hookrightarrow}[\mathcal{L}(\mathcal{A}, q)]$, while the latter state q keeps accepting terms only in $\mathcal{L}(\mathcal{A}, q)$.

Example 11. Let $\mathcal{R} = (R, \mu)$ where $R = \{a \to b, f(x) \to g(x)\}$ and $\mu(f) = \emptyset, \mu(g) = \{1\}$. Let $\mathcal{A}$ a TA that recognizes $\{f(a)\}$ defined by a final state $\{q_{fa}\}$ and transition rules $\{a \to q_a, f(q_a) \to q_{fa}\}$. The procedure produces the following TA defined by final states: $Q_*^f = \{\tilde{q}_{fa}\}$, and transition rules: $\Delta_* = \{a \to q_a, a \to \tilde{q}_a, f(q_a) \to q_{fa}, f(q_a) \to \tilde{q}_{fa}, b \to \tilde{q}_a, g(\tilde{q}_a) \to \tilde{q}_{fa}\}$.

The TA $\mathcal{A}_*$ accepts terms $f(a)$, $g(a)$ and $g(b)$ in $\underset{\mathcal{R}}{\hookrightarrow}[\{f(a)\}]$, and does not accept $f(b)$. □

We show the procedure $\mathrm{P_{cs}}$ in Figure 4, where we use a notation $\widetilde{Q}$ to represent $\{\tilde{q} \mid q \in Q\}$, and q^* to represent either q or $\tilde{q}$.

Example 12. Let us follow how procedure $\mathrm{P_{cs}}$ works. Consider R and $\mathcal{A}$ in Example 11.

In the initializing step, we have $\Delta_{\mathrm{RS}} = \emptyset$, $Q_* = \{q_a, q_{fa}, \tilde{q}_a, \tilde{q}_{fa}\}$, $Q_*^f = \{\tilde{q}_{fa}\}$, $\Delta_0 = \Delta \cup \{a \to \tilde{q}_a, f(q_a) \to \tilde{q}_{fa}\}$.

The saturation steps stop at $k = 1$, we have $\Delta_1 = \Delta_0 \cup \{b \to \tilde{q}_a, g(\tilde{q}_a) \to \tilde{q}_{fa}\}$, $\Delta_2 = \Delta_1$. □

Input TA $\mathcal{A} = \langle F, Q, Q^f, \Delta \rangle$ and right-shallow CS-TRS $\mathcal{R} = (R, \mu)$ over F.

Output TA $\mathcal{A}_* = \langle F, Q_*, Q_*^f, \Delta_* \rangle$ such that $\mathcal{L}(\mathcal{A}_*) = \underset{\mathcal{R}}{\hookrightarrow}[\mathcal{L}(\mathcal{A}_*)]$, if R is linear.

Step 1 (initialize) 1. Prepare a TA $\mathcal{A}_{\mathrm{RS}} = \langle F, Q_{\mathrm{RS}}, Q_{\mathrm{RS}}^f, \Delta_{\mathrm{RS}} \rangle$ that recognizes $\mathrm{RS}(R)$ (the same as the procedure $\mathrm{P_{in}}$). Here we assume $Q_{\mathrm{RS}}^f = \{q^t \mid t \in \mathrm{RS}(R)\}$ and $\mathcal{L}(\mathcal{A}_{\mathrm{RS}}, q^t) = \{t\}$.

2. Let
   - $k := 0$
   - $Q_* = (Q \uplus Q_{\mathrm{RS}}) \cup (\widetilde{Q \uplus Q_{\mathrm{RS}}})$
   - $Q_*^f = \widetilde{Q^f}$
   - $\Delta_0 = \Delta \cup \{\tilde{q}' \to \tilde{q} \mid q' \to q \in \Delta\}$

$$\cup \left\{ f(p_i, \ldots, p_n) \to \tilde{q} \;\middle|\; \begin{array}{l} f(q_1, \ldots, q_n) \to q \in \Delta, \\ p_i = \begin{cases} \tilde{q}_i & \cdots \text{ if } i \in \mu(f), \\ q_i & \cdots \text{ otherwise} \end{cases} \end{array} \right\}$$

Step 2 Let Δ_{k+1} be transition rules produced by augmenting transition rules of Δ_k by following inference rules:

$$\frac{f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R \quad f(q_1^*, \ldots, q_n^*) \to \tilde{q} \in \Delta_k}{g(q_1'^*, \ldots, q_m'^*) \to \tilde{q} \in \Delta_{k+1}}$$

if there exists $\theta : X \to Q_*$ such that

- $l_i\theta \xrightarrow[\Delta_k]{*} q_i^*$ for all $1 \leq i \leq n$,
- $q_j'^* = \begin{cases} \tilde{p}_j & \cdots j \in \mu(g), r_j \in X,\ p_j^* = r_j\theta \\ p_j^* & \cdots j \notin \mu(g), r_j \in X,\ p_j^* = r_j\theta \\ \tilde{q}^{r_j} & \cdots j \in \mu(g), r_j \notin X \\ q^{r_j} & \cdots j \notin \mu(g), r_j \notin X \end{cases}$ for all $1 \leq j \leq m$

and

$$\frac{f(l_1, \ldots, l_n) \to x \in R \quad f(q_1^*, \ldots, q_n^*) \to \tilde{q} \in \Delta_k}{\tilde{q}' \to \tilde{q} \in \Delta_{k+1}}$$

if there exists $\theta : X \to Q_*$ such that

- $l_i\theta \xrightarrow[\Delta_k]{*} q_i^*$ for all $1 \leq i \leq n$, and
- $\tilde{q}' = \tilde{p}$ where $p^* = x\theta$.

Step 3 If $\Delta_{k+1} = \Delta_k$ then stop and set $\Delta_* = \Delta_k$; Otherwise $k := k+1$ and goto step 2.

Fig. 4. Procedure $\mathrm{P_{cs}}$

The procedure P_{cs} eventually terminates at some k, because rewrite rules in R and states Q_* are finite and hence possible transitions rules are finite. Apparently $\Delta_0 \subset \cdots \subset \Delta_k = \Delta_{k+1} = \cdots$.

We show several technical lemmas.

Proposition 13. *If* $t \xrightarrow[\Delta_k]{*} q \in Q \uplus Q_{RS}$, *then* $t \xrightarrow[\Delta_0]{*} q$.

Proof. The proposition follows from the fact that transition rules having $q \in Q \uplus Q_{\mathrm{RS}}$ on right-hand sides are in Δ or Δ_{RS}. □

Proposition 14. $t \xrightarrow[\Delta_0]{*} \tilde{q} \in \widetilde{Q \uplus Q_{RS}}$ *if and only if* $t \xrightarrow[\Delta_0]{*} q \in Q \uplus Q_{RS}$.

Proof. From construction of Δ_0. □

Proposition 15. *If* $t \xrightarrow[\Delta_k]{*} q \in Q \uplus Q_{RS}$, *then* $t \xrightarrow[\Delta_k]{*} \tilde{q}$.

Proof. Let $t \xrightarrow[\Delta_k]{*} q$, then $t \xrightarrow[\Delta_0]{*} q$ by Proposition 13. The lemma follows from Proposition 14 and $\Delta_0 \subseteq \Delta_k$. □

Lemma 16. *If* $t[t']_p \xrightarrow[\Delta_k]{*} \tilde{q}$ *and* $p \in \mathrm{Pos}^{\mu}(t)$, *then there exists* $\tilde{q}'$ *such that* $t' \xrightarrow[\Delta_k]{*} \tilde{q}'$ *and* $t[\tilde{q}']_p \xrightarrow[\Delta_k]{*} \tilde{q}$.

Proof. We show it by induction on the length n of transition sequence $\alpha : t[t']_p \xrightarrow[\Delta_k]{n} \tilde{q}$. Let $t[t']_p \xrightarrow[\Delta_k]{n} \tilde{q}$ and $p \in \mathrm{Pos}^{\mu}(t)$.

1. If $p = \varepsilon$, then $t = t'$, and hence $t' \xrightarrow[\Delta_k]{*} \tilde{q}$ follows.
2. Consider the case $p = ip'$ for some $i \in \mathbb{N}$. Then α can be represented as $t[t']_p = f(\ldots, t_{i-1}, t_i[t']_{p'}, t_{i+1}, \ldots) \xrightarrow[\Delta_k]{n-1} f(\ldots, q^*_{i-1}, q^*_i, q^*_{i+1}, \ldots) \xrightarrow[\Delta_k]{} \tilde{q}$. Since $ip' = p \in \mathrm{Pos}^{\mu}(t)$, we have $i \in \mu(f)$. Hence $q^*_i = \tilde{q}_i$ follows from the construction of Δ_k.
   By the induction hypothesis, there exists $\tilde{q}'$ such that $t' \xrightarrow[\Delta_k]{*} \tilde{q}'$ and $t_i[\tilde{q}']_{p'} \xrightarrow[\Delta_k]{*} \tilde{q}_i$. Here we have $t[\tilde{q}']_p = f(\ldots, t_{i-1}, t_i[\tilde{q}']_{p'}, t_{i+1}, \ldots) \xrightarrow[\Delta_k]{*} f(\ldots, q^*_{i-1}, \tilde{q}_i, q^*_{i+1}, \ldots) \xrightarrow[\Delta_k]{} \tilde{q}$. □

The following lemma is obtained from the above propositions and lemmas.

Lemma 17. *Let* $\mathcal{R}$ *be left-linear and right-shallow. Then* $s \xrightarrow[\Delta_*]{*} \tilde{q}$ *and* $s \overset{n}{\underset{\mathcal{R}}{\hookrightarrow}} t$ *imply* $t \xrightarrow[\Delta_*]{*} \tilde{q}$.

Proof. We present the proof in the case $n = 1$. Let $s \xrightarrow[\Delta_k]{*} \tilde{q}$ and $s = s[l\sigma]_p \underset{\mathcal{R}}{\hookrightarrow} s[r\sigma]_p = t$ for some rewrite rule $l \to r \in R$, where $p \in \mathrm{Pos}^{\mu}(s)$. We have a transition sequence $s \xrightarrow[\Delta_k]{*} s[\tilde{q}']_p \xrightarrow[\Delta_k]{*} \tilde{q}$ by Lemma 16.

1. Consider the case where the rewrite rule is in the form $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m)$. Since this rewrite rule is left-linear, $s \xrightarrow[\Delta_k]{*} \tilde{q}$ is represented as $s = s[f(l_1\sigma, \ldots, l_n\sigma)]_p \xrightarrow[\Delta_k]{*} s[f(l_1\theta, \ldots, l_n\theta)]_p \xrightarrow[\Delta_k]{*} s[f(q^*_1, \ldots, q^*_n)]_p \xrightarrow[\Delta_k]{}$

$s[\tilde{q}']_p \xrightarrow[\Delta_k]{*} \tilde{q}$ for some $\theta : X \to Q_*$. Here we have $l_i\theta \xrightarrow[\Delta_k]{*} q_i^*$. From the construction of $\mathcal{A}_*$, there exists a transition rule $g(q_1'^*, \ldots, q_n'^*) \to \tilde{q}' \in \Delta_{k+1}$ such that

$$q_j'^* = \begin{cases} \tilde{p}_j & \cdots j \in \mu(g), r_j \in X \\ p_j^* & \cdots j \notin \mu(g), r_j \in X \\ \tilde{q}^{r_j} & \cdots j \in \mu(g), r_j \notin X \\ q^{r_j} & \cdots j \notin \mu(g), r_j \notin X \end{cases}$$

where $p_j^* = r_j\theta$.

(a) For j such that $r_j \in X$, we have $l_i|_p = r_j$ for some i and p. Hence $r_j\sigma = l_i|_p\sigma \xrightarrow[\Delta_k]{*} l_i|_p\theta = r_j\theta = p_j^*$. Since we have $r_j\sigma \xrightarrow[\Delta_k]{*} \tilde{p}_j$ by Proposition 15, we obtain $r_j\sigma \xrightarrow[\Delta_k]{*} q_j'^*$ in either case of $q_j'^* = \tilde{p}_j$ or $q_j'^* = p_j$.

(b) For j such that $r_j \notin X$, we have $r_j\sigma = r_j$ from the shallowness of r_j and also have $r_j \xrightarrow[\Delta_0]{*} q^{r_j}$. Thus $r_j\sigma \xrightarrow[\Delta_0]{*} q^{r_j}$. Since we have $r_j\sigma \xrightarrow[\Delta_0]{*} \tilde{q}^{r_j}$ by Proposition 15, we obtain $r_j\sigma \xrightarrow[\Delta_k]{*} q_j'^*$ in either case of $q_j'^* = \tilde{q}^{r_j}$ or $q_j'^* = q^{r_j}$.

Therefore we have $t = s[g(r_1, \ldots, r_m)\sigma]_p \xrightarrow[\Delta_k]{*} s[g(q_1'^*, \ldots, q_m'^*)]_p \xrightarrow[\Delta_{k+1}]{} s[\tilde{q}']_p \xrightarrow[\Delta_k]{*} \tilde{q}$.

2. In the case where the rewrite rule is in the form $f(l_1, \ldots, l_n) \to x$, we can show it similarly to the previous case. □

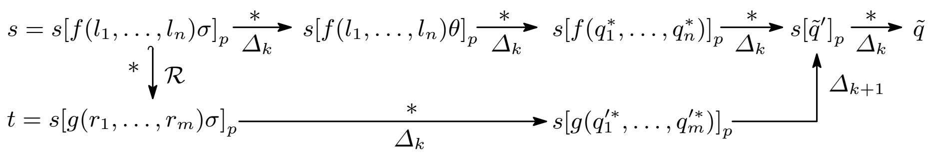

Fig. 5. The diagram of proof of lemma 17

Lemma 18. *If $\mathcal{R}$ is left-linear and right-shallow then $\mathcal{L}(\mathcal{A}_*) \supseteq \underset{\mathcal{R}}{\hookrightarrow}[\mathcal{L}(\mathcal{A})]$.*

Proof. Let $s \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} t$ and $s \xrightarrow[\Delta]{*} q \in Q^f$. Since $s \xrightarrow[\Delta_0]{*} q$ from construction of Δ_0, we have $s \xrightarrow[\Delta_0]{*} \tilde{q}$ by Proposition 15. Hence $t \xrightarrow[\Delta_*]{*} \tilde{q} \in Q_*^f$ by lemma17. □

Lemma 19. *Let Δ_* be generated from linear right-shallow CS-TRS. Then*

1. *$\alpha : t = g(t_1, \ldots, t_m) \xrightarrow[\Delta_*]{*} g(q_1'^*, \ldots, q_m'^*) \xrightarrow[\Delta_{k+1} \setminus \Delta_k]{} \tilde{q}$ where $t_j \xrightarrow[\Delta_0]{*} q_j'^*$ for all $j \in \mu(g)$, or*
2. *$\alpha : t \xrightarrow[\Delta_0]{*} \tilde{q}' \xrightarrow[\Delta_{k+1} \setminus \Delta_k]{} \tilde{q}$,*

implies $s \underset{\mathcal{R}}{\hookrightarrow} t$, $\beta : s \xrightarrow[\Delta_*]{*} \tilde{q}$ and $\alpha \sqsupset \beta$ for some term s.

Proof. Consider the first case. Since the last transition rule applied in α is introduced by the first inference rule in the procedure, there exist $f(l_1,\ldots,l_n) \to g(r_1,\ldots,r_m) \in R$, $f(q_1^*,\ldots,q_n^*) \to \tilde{q} \in \Delta_k$, and $\theta : X \to Q_*$ such that

- $l_i\theta \xrightarrow[\Delta_k]{*} q_i^*$ for all $1 \le i \le n$,
- $q_j'^* = \begin{cases} \tilde{p}_j & \cdots j \in \mu(g), r_j \in X,\ p_j^* = r_j\theta \\ p_j^* & \cdots j \notin \mu(g), r_j \in X,\ p_j^* = r_j\theta \\ \tilde{q}^{r_j} & \cdots j \in \mu(g), r_j \notin X \\ q^{r_j} & \cdots j \notin \mu(g), r_j \notin X \end{cases}$ for all $1 \le j \le m$, and
- $\mathcal{L}(\Delta_0, x\theta) \neq \emptyset$ for each erasing variable x.

We have the following:

1. For $j \in \mu(g)$ such that $r_j \in X$, we have $q_j'^* = \tilde{p}_j$ and $t_j \xrightarrow[\Delta_0]{*} q_j'^*$. Hence we have $t_j \xrightarrow[\Delta_0]{*} r_j\theta$ by Proposition 14.
2. For $j \notin \mu(g)$ such that $r_j \in X$, we have $t_j \xrightarrow[\Delta_*]{*} q_j'^* = r_j\theta$.
3. For $j \in \mu(g)$ such that $r_j \notin X$, we have $t_j \xrightarrow[\Delta_0]{*} q_j'^* = \tilde{q}^{r_j}$. Since $t_j \xrightarrow[\Delta_0]{*} q^{r_j}$ by Proposition 14, we have $t_j = r_j$ from the construction of Δ_0. Therefore $t_j = r_j\theta$ follows from right-shallowness.
4. For $j \notin \mu(g)$ such that $r_j \notin X$, we have $t_j \xrightarrow[\Delta_*]{*} q_j'^* = q^{r_j}$. Since $t_j \xrightarrow[\Delta_0]{*} q^{r_j}$ by Proposition 13, we have $t_j = r_j$ from the construction of Δ_0. Therefore $t_j = r_j\theta$ follows from right-shallowness.

Thus we have $g(t_1,\ldots,t_m) \xrightarrow[\Delta_*]{*} g(r_1\theta,\ldots,r_m\theta) \xrightarrow[\Delta_0]{*} g(q_1'^*,\ldots,q_m'^*)$.

We define a substitution $\sigma :\ \mathrm{Var}(f(l_1,\ldots,l_n)) \to \mathcal{T}(F)$ as follows:

$$x\sigma = \begin{cases} t_j & \cdots \text{ if there exists } j \text{ such that } r_j = x \\ t' & \cdots \text{ otherwise, choose an arbitral } t' \text{ such that } t' \xrightarrow[\Delta_0]{*} x\theta, \end{cases}$$

where σ is well-defined from the right-linearity of rewrite rules. We can construct $\beta : f(l_1,\ldots,l_n)\sigma \xrightarrow[\Delta_*]{*} f(l_1,\ldots,l_n)\theta \xrightarrow[\Delta_k]{*} f(q_1^*,\ldots,q_n^*) \xrightarrow[\Delta_k]{} \tilde{q}$.

On the other hand, we have $f(l_1,\ldots,l_n)\sigma \xhookrightarrow[\mathcal{R}]{} g(r_1,\ldots,r_m)\sigma = g(t_1,\ldots,t_m) = t$. Therefore the lemma of this case follows by taking $f(l_1,\ldots,l_n)\sigma$ as s. Here $\alpha \sqsupset \beta$ follows from the left-linearity of rewrite rules.

For the case where the transition rule applied last in α is (in the form of) $\tilde{q}' \to \tilde{q} \in \Delta_{k+1} \setminus \Delta_k$, the lemma can be shown in similar to the previous case. □

Lemma 20. *If $\mathcal{R}$ be linear and right-shallow then $\mathcal{L}(\mathcal{A}_*) \subseteq \xhookrightarrow[\mathcal{R}]{}[\mathcal{L}(\mathcal{A})]$.*

Proof. Let $t \xrightarrow[\Delta_*]{*} \tilde{q} \in Q_*^f$ and the following claim holds:

$$\alpha :\ t \xrightarrow[\Delta_*]{*} \tilde{q} \text{ implies } s \xrightarrow[\Delta_0]{*} \tilde{q} \text{ and } s \xhookrightarrow[\mathcal{R}]{*} t.$$

Then, we have $s \xrightarrow[\Delta_0]{*} \tilde{q}$ and $s \xhookrightarrow[\mathcal{R}]{*} t$ for some s. Hence $s \in \mathcal{L}(\mathcal{A})$ follows from Proposition 14.

In the sequel, we prove the claim by induction on α with respect to $\sqsupset$.

1. Consider the case that the last transition rule applied in α is (in the form of) $g(q_1'^*, \ldots, q_m'^*) \to \tilde{q} \in \Delta_k$. Then α can be represented as $t = g(t_1, \ldots, t_m) \xrightarrow[\Delta_*]{*} g(q_1'^*, \ldots, q_m'^*) \xrightarrow[\Delta_k]{} \tilde{q}$.
   (a) For $j \in \mu(g)$ such that $q_j'^* = \tilde{q}_j'$, there exists s_j such that $s_j \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \tilde{q}_j' = q_j'^*$ from the induction hypothesis, since $\alpha \sqsupset (t_j \xrightarrow[\Delta_*]{*} \tilde{q}_j')$.
   (b) For $j \notin \mu(g)$ such that $q_j'^* = \tilde{q}_j'$, we take s_j as t_j.
   (c) For j such that $q_j'^* = q_j'$, we have $t_j \xrightarrow[\Delta_0]{*} q_j' = q_j'^*$ by Proposition 14. We take s_j as t_j.

   Now we have $g(s_1, \ldots, s_m) \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} g(t_1, \ldots, t_m) = t$ and $\alpha' : g(s_1, \ldots, s_m) \xrightarrow[\Delta_*]{*} g(q_1'^*, \ldots, q_m'^*,) \xrightarrow[\Delta_k]{} \tilde{q}$, where $\alpha \sqsupseteq \alpha'$ and $s_j \xrightarrow[\Delta_0]{*} q_j'^*$ for all $j \in \mu(g)$.
   In the subcase $k = 0$ we have no j that satisfies (b), since $j \in \mu(g)$ if and only if $q_j'^* = \tilde{q}_j'$ from the construction of Δ_0. Thus every transition rule used in α' is in Δ_0. Therefore the claim trivially holds by letting $s = g(s_1, \ldots, s_m)$.
   In the subcase $k > 0$, we have $s' \underset{\mathcal{R}}{\hookrightarrow} g(s_1, \ldots, s_m)$ and $\beta : s' \xrightarrow[\Delta_*]{*} \tilde{q}$ for some s' such that $\alpha' \sqsupset \beta$ by Lemma 19. Therefore the claim holds by the induction hypothesis since $\alpha \sqsupset \beta$.
2. In the case where the last transition rule applied in α is (in the form of) $q'^* \to q^* \in \Delta_k$, we can show it similarly to the previous case. $\square$

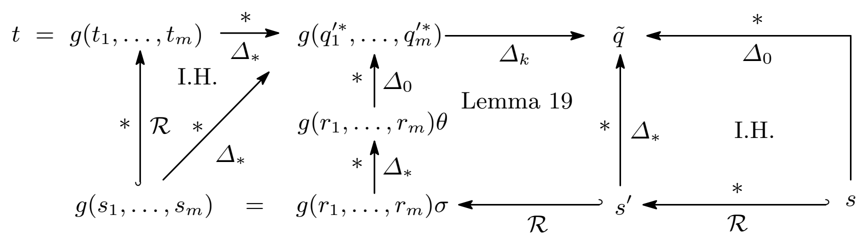

Fig. 6. The diagram of proof of lemma 20

The following theorem is proved by lemma 18, lemma 20 and theorem 1.

Theorem 21. *Context-sensitive reduction for linear right-shallow TRSs effectively preserves recognizability. Thus context-sensitive reachability is decidable for linear right-shallow TRSs.*

5 Discussion

The authors think that the left-linear restriction for the context-sensitive case will be removed by modifying the procedure P_{cs} similar to [11] if all variables that occur in $\mathrm{Pos}(r) \setminus \mathrm{Pos}^{\mu}(r)$ are left-linear. For the innermost case, similar modification may be possible. However, constructing $\mathcal{A}_{\mathrm{NF}}$ would be a barrier. Using automata with an equality test between brothers [1] is a possible direction

On the other hand, removing the right-linearity restriction is impossible for both cases, because there exists a counter example such as the following.

Example 22. For TRS $R = \{g(x) \to f(x,x)\}$ and a regular set $G = \{g(t) \mid t \in T(\{a,h\})\}$, $\underset{R}{\to}[G] = \underset{R}{\to}_{\mathrm{in}}[G] = \{g(t), f(t,t) \mid t \in T(\{a,h\})\}$ is not a regular. □

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