Proving Confluence of Conditional Term Rewriting Systems via Unravelings *

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Abstract

Unravelings are a class of transformations of conditional term rewriting systems into unconditional systems. Such transformations have been used to analyze and simulate conditional rewrite steps by unconditional rewrite steps for properties like (operational) termination. In this paper, we show how to prove confluence of conditional term rewriting systems via unravelings.

1 Introduction and Overview

Conditional term rewriting systems (CTRSs) are term rewriting systems in which rules may be constrained by equations over terms. Such systems arise naturally in many settings like functional programming and they have been used in applications like program inversion [8].

Yet, CTRSs are more difficult to analyze and many criteria that hold for unconditional TRSs do not hold for CTRSs. Therefore, several transformations have been defined that eliminate conditions in CTRSs [6, 14, 1, 12].

There are some results on confluence of CTRSs like [2, 13], yet there are no results known to us that use transformations to prove confluence of CTRSs.

The main difficulty in using transformations to prove confluence is that the transformed TRS may give rise to derivations that are not possible in the original CTRS. Another difficulty is that in order to encode conditions, the signature of the transformed system is different from the signature of the original CTRS. This might lead to derivations in which terms occur that are not defined in the original system.

In this paper, we show how to prove confluence of CTRSs via *unravelings*, the simplest class of transformations of CTRSs into TRSs. We focus on so-called *oriented, deterministic 3-CTRSs*, a class of CTRSs in which extra variables are allowed to a certain extent. We will use common notions and notations, like they are used in e.g. [10].

2 Unravelings

Unravelings are a class of transformations of CTRSs into unconditional TRSs that have been introduced in [6]. They have been the subject of interest in several publications [8, 4, 9, 5].

In an unraveling, a conditional rule is split into several unconditional rules. The conditions are encoded in new function symbols, called *U-symbols*, along with some variables. If the conditions are satisfied, then the rhs of the original conditional rule is reproduced.

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In the unraveling $\mathbb{U}_{seq}$, that is defined in [10] (based on [7]), one new function symbol is introduced for each condition in a conditional rule. By sequentially encoding the conditions, this unraveling can transform deterministic CTRSs (DCTRSs) into TRSs without extra variables. In DCTRSs, extra variables must occur on the rhs of a condition first so that their matchers can be determined by plain rewriting.

A conditional rule $\alpha : l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k$ is transformed into unconditional rules as follows:

$$\mathbb{U}_{seq}(\alpha) = \{ \ l \to U_1^\alpha(s_1, \vec{X}_1), \ \ U_1^\alpha(t_1, \vec{X}_1) \to U_2^\alpha(s_2, \vec{X}_2), \ \ \ldots, \ \ U_k^\alpha(t_k, \vec{X}_k) \to r \ \}$$

where $X_i = \mathcal{V}ar(l, t_1, \ldots, t_{i-1})$. Here, $\vec{X}$ denotes the unique sequence of variables in X under some fixed order on variables.

In order to distinguish terms in the unraveling that contain U-symbols, we will refer to such terms as *mixed terms* while we refer to terms without U-symbols as *original terms*.

It is easy to show that a derivation in a CTRS $u \to_{\mathcal{R}}^* v$ has a corresponding derivation in the unraveling $u \to_{\mathbb{U}_{seq}(\mathcal{R})}^* v$ (for all original terms u, v). This property is called *completeness*. While completeness is easy to prove and satisfied in general, its counterpart *soundness* only holds in certain cases (see e.g. [6, 4, 9, 5]).

3 Soundness for Joinability

We prove confluence of a CTRS $\mathcal{R}$ via derivations in the unraveling of $\mathcal{R}$ in the following way: for all original terms s, t_1, t_2 such that $t_1 \leftarrow_{\mathcal{R}}^* s \to_{\mathcal{R}}^* t_2$, we know by completeness that $t_1 \leftarrow_{\mathbb{U}(\mathcal{R})}^* s \to_{\mathbb{U}(\mathcal{R})}^* t_2$; if there is an original term u such that $t_1 \to_{\mathbb{U}(\mathcal{R})}^* u \leftarrow_{\mathbb{U}(\mathcal{R})}^* t_2$, then by soundness we obtain that $t_1 \to_{\mathcal{R}}^* u \leftarrow_{\mathcal{R}}^* t_2$ in the original CTRS $\mathcal{R}$.

One difficulty in this approach is that we need to prove that t_1 and t_2 have a common descendant in $\mathbb{U}(\mathcal{R})$ that is an original term. We will therefore use another notion of soundness:

Definition 1 (Soundness for joinability)**.** *An unraveling* $\mathbb{U}$ *is* sound for joinability *of a CTRS* $\mathcal{R}$ *if for all original terms* s, t *such that* $s \downarrow_{\mathbb{U}(\mathcal{R})} t$*, also* $s \downarrow_{\mathcal{R}} t$*.*

We can use soundness for joinability to prove confluence for every CTRS:

Lemma 2. *Let* $\mathcal{R}$ *be a DCTRS and* $\mathbb{U}$ *be an unraveling. If* $\mathbb{U}(\mathcal{R})$ *is confluent and* $\mathbb{U}$ *is sound for joinability of* $\mathcal{R}$*, then* $\mathcal{R}$ *is confluent.*

Proof. Consider two terms s, t such that $s \leftrightarrow_{\mathcal{R}}^* t$. Completeness of $\mathbb{U}$ implies $s \leftrightarrow_{\mathbb{U}(\mathcal{R})}^* t$. Since $\mathbb{U}(\mathcal{R})$ is confluent, therefore $s \downarrow_{\mathbb{U}(\mathcal{R})} t$, and by soundness of joinability $s \downarrow_{\mathcal{R}} t$. ☐

Although there is a strong connection between soundness and soundness for joinability, soundness does not imply soundness for joinability in general:

Example 3. Consider the following CTRS $\mathcal{R}$ that contains one conditional rule that is unraveled into two unconditional rules (using $\mathbb{U}_{seq}$):

$$\mathbb{U}_{seq}(\left\{ \begin{array}{l} \quad\ \mathsf{a} \to \mathsf{c} \to \mathsf{e} \\ \quad\ \ \ \boxtimes \ \ \searrow \\ \quad\ \mathsf{b} \to \mathsf{d} \to \mathsf{k} \\ \quad \mathsf{f}(x) \to x \Leftarrow x \to^* \mathsf{e} \\ \mathsf{g}(x,x) \to \mathsf{h}(x,x) \\ \mathsf{h}(\mathsf{d},x) \to \mathsf{A}(x) \end{array} \right\}) = \left\{ \begin{array}{ll} \quad\ \vdots & \\ \mathsf{f}(x) \to U_1^\alpha(x,x), & U_1^\alpha(\mathsf{e},x) \ \to x \\ \quad\ \vdots & \end{array} \right\}$$

$\mathbb{U}_{seq}$ is sound for non-erasing 2-DCTRSs ([5, Theorem 18] and [9, Corollary 5.5]), therefore the unraveling is sound. In $\mathbb{U}_{seq}(\mathcal{R})$, terms $\mathsf{g}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))$ and $\mathsf{A}(\mathsf{f}(\mathsf{k}))$ are joinable:

$$\mathsf{g}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b})) \to^* \mathsf{h}(U_1^\alpha(\mathsf{c},\mathsf{d}),U_1^\alpha(\mathsf{c},\mathsf{d})) \to^* \mathsf{h}(\mathsf{d},U_1^\alpha(\mathsf{c},\mathsf{d})) \to \mathsf{A}(U_1^\alpha(\mathsf{c},\mathsf{d})) \to^* \mathsf{A}(U_1^\alpha(\mathsf{k},\mathsf{k})) \leftarrow \mathsf{A}(\mathsf{f}(\mathsf{k}))$$

In $\mathcal{R}$, $\mathsf{A}(\mathsf{f}(\mathsf{k}))$ is irreducible, therefore $\mathsf{g}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b})) \downarrow_{\mathcal{R}} \mathsf{A}(\mathsf{f}(\mathsf{k}))$ only if $\mathsf{g}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b})) \to^*_{\mathcal{R}} \mathsf{A}(\mathsf{f}(\mathsf{k}))$. Yet, for this we need some common reduct s of $\mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b})$ such that $s \to^*_{\mathcal{R}} \mathsf{d}$ and $s \to^*_{\mathcal{R}} \mathsf{f}(\mathsf{k})$, but there is no original term satisfying these properties.

In the previous example, the derivation in $\mathbb{U}_{seq}(\mathcal{R})$ contains mixed terms. In order to prove soundness for joinability, we use a mapping t that translates terms in $\mathbb{U}(\mathcal{R})$ (including mixed terms) into original terms. Using such a translation we obtain a more general soundness criterion: an unraveling $\mathbb{U}$ is sound w.r.t. t for a DCTRS $\mathcal{R}$, if for all original terms u and all mixed terms v' such that $u \to^*_{\mathbb{U}(\mathcal{R})} v'$, $\mathsf{t}(v')$ is defined and $u \to^*_{\mathcal{R}} \mathsf{t}(v')$. Soundness w.r.t. t implies soundness for joinability:

Lemma 4. *If an unraveling $\mathbb{U}$ is sound w.r.t. t for a DCTRS $\mathcal{R}$, then $\mathbb{U}$ is also sound for joinability of $\mathcal{R}$.*

Proof. Let s,t be two original terms such that there is some (possibly mixed) term u' such that $s \to^*_{\mathbb{U}(\mathcal{R})} u' \leftarrow^*_{\mathbb{U}(\mathcal{R})} t$. Then, soundness w.r.t. t implies $s \to^*_{\mathcal{R}} \mathsf{t}(u') \leftarrow^*_{\mathcal{R}} t$. □

4 A New Unraveling

For many CTRSs, in particular overlay CTRSs, $\mathbb{U}_{seq}$ returns a non-confluent TRS so that we cannot use $\mathbb{U}_{seq}$ to prove confluence of the original CTRSs.

Example 5 ([11])**.** The following CTRS defines *even* and *odd* predicates for natural number encoded by $\mathsf{0}$ and s:

$$\mathcal{R}_{\mathsf{even}} = \left\{ \begin{array}{cc} \mathsf{even}(\mathsf{0}) \to \mathsf{true} & \mathsf{odd}(\mathsf{0}) \to \mathsf{false} \\ \mathsf{even}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{odd}(x) \to^* \mathsf{true} & \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{even}(x) \to^* \mathsf{true} \\ \mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{odd}(x) \to^* \mathsf{false} & \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{even}(x) \to^* \mathsf{false} \end{array} \right\}$$

The CTRS is unraveled into the following TRS using $\mathbb{U}_{seq}$:

$$\mathbb{U}_{seq}(\mathcal{R}_{\mathsf{even}}) = \left\{ \begin{array}{ll} \mathsf{even}(\mathsf{0}) \to \mathsf{true} & \mathsf{odd}(\mathsf{0}) \to \mathsf{false} \\ \mathsf{even}(\mathsf{s}(x)) \to U_1^\alpha(\mathsf{odd}(x),x) & \mathsf{odd}(\mathsf{s}(x)) \to U_1^\gamma(\mathsf{even}(x),x) \\ U_1^\alpha(\mathsf{true},x) \to \mathsf{false} & U_1^\gamma(\mathsf{true},x) \to \mathsf{false} \\ \mathsf{even}(\mathsf{s}(x)) \to U_1^\beta(\mathsf{odd}(x),x) & \mathsf{odd}(\mathsf{s}(x)) \to U_1^\eta(\mathsf{even}(x),x) \\ U_1^\beta(\mathsf{false},x) \to \mathsf{true} & U_1^\eta(\mathsf{false},x) \to \mathsf{true} \end{array} \right\}$$

The unraveled TRS is not confluent, for instance $\mathsf{even}(\mathsf{s}(\mathsf{0}))$ rewrites to $U_1^\alpha(\mathsf{odd}(\mathsf{s}(\mathsf{0}),\mathsf{0})$ and $U_1^\beta(\mathsf{odd}(\mathsf{s}(\mathsf{0}),\mathsf{0})$ that are not joinable. Note that a more complicated and practical example with the non-confluence problem can be found in [10, Example 7.2.49].

The following new unraveling returns a confluent TRS for certain overlay CTRSs. It strongly resembles the unraveling $\mathbb{U}_{seq}$, but we introduce new U-symbols based on the lhs of the transformed rule and terms in the conditions:

Definition 6 (New unraveling)**.** *Let α be a conditional rule $l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k$ $(k \geq 0)$, then its unraveling is defined as*

$$\mathbb{U}_{conf}(\alpha) = \left\{ \begin{array}{r} l \to U_{l,s_1}(s_1, \vec{X}_1) \\ U_{l,s_1}(t_1, \vec{X}_1) \to U_{l,s_1,t_1,s_2}(s_2, \vec{X}_2) \\ \vdots \\ U_{l,s_1,t_1,\ldots,t_{k-1},s_k}(t_k, \vec{X}_k) \to r \end{array} \right\}$$

where $X_i = \mathcal{V}ar(l, t_1, \ldots, t_{i-1})$. Note that for U_α and $U_{\alpha'}$, we use the same symbol, e.g., U_α, if α' is a renamed variant of U_α. The unraveled TRS $\mathbb{U}_{conf}(\mathcal{R})$ then is $\bigcup_{\alpha \in \mathcal{R}} \mathbb{U}_{conf}(\alpha)$.

In order to prove soundness for joinability of certain cases, we use the following backtranslation that is also used in [5]:

$$\begin{array}{rll} \mathsf{tb}(x) &= x & \text{for all variables } x \\ \mathsf{tb}(U_{l,\ldots}(v, \vec{X}_i\sigma)) &= l\,\mathsf{tb}(\sigma) & \text{for all U-symbols } U_{l,\ldots} \\ \mathsf{tb}(f(t_1, \ldots, t_{ar(f)})) &= f(\mathsf{tb}(t_1), \ldots, \mathsf{tb}(t_{ar(f)})) & \text{for all non-U-symbols } f \end{array}$$

$\mathbb{U}_{seq}$ is sound w.r.t. tb for weakly left-linear CTRSs and since tb is well-defined for $\mathbb{U}_{conf}$ we can adapt the proof of [5, Theorem 3.28] to $\mathbb{U}_{conf}$.

Lemma 7. *$\mathbb{U}_{conf}$ is sound w.r.t.* tb *for a weakly left-linear CTRSs.*

Proof (Sketch). Since tb is well-defined and derivations in the conditions can be extracted from the U-symbols, we can use the proof of [5, Theorem 3.28]. ☐

Corollary 8. *$\mathbb{U}_{conf}$ is sound for joinability of weakly left-linear CTRSs.*

Finally, we obtain our main result:

Theorem 9. *A weakly left-linear DCTRS $\mathcal{R}$ is confluent if so is $\mathbb{U}_{conf}(\mathcal{R})$.*

Example 10. Consider the CTRS of Example 5. Its unraveling for $\mathbb{U}_{conf}$ has two rule less:

$$\mathbb{U}_{conf}(\mathcal{R}_{\mathsf{even}}) = \left\{ \begin{array}{r} \mathsf{even}(0) \to \mathsf{true} \\ \mathsf{even}(\mathsf{s}(x)) \to U_{\mathsf{even}(\mathsf{s}(x)),\mathsf{odd}(x)}(\mathsf{odd}(x), x) \\ U_{\mathsf{even}(\mathsf{s}(x)),\mathsf{odd}(x)}(\mathsf{true}, x) \to \mathsf{false} \\ U_{\mathsf{even}(\mathsf{s}(x)),\mathsf{odd}(x)}(\mathsf{false}, x) \to \mathsf{true} \\ \mathsf{odd}(0) \to \mathsf{false} \\ \mathsf{odd}(\mathsf{s}(x)) \to U_{\mathsf{odd}(\mathsf{s}(x)),\mathsf{even}(x)}(\mathsf{even}(x), x) \\ U_{\mathsf{odd}(\mathsf{s}(x)),\mathsf{even}(x)}(\mathsf{true}, x) \to \mathsf{false} \\ U_{\mathsf{odd}(\mathsf{s}(x)),\mathsf{even}(x)}(\mathsf{false}, x) \to \mathsf{true} \end{array} \right\}$$

The unraveled TRS is now confluent. It follows from left-linearity of $\mathbb{U}_{seq}(\mathcal{R}_{\mathsf{even}})$ that $\mathcal{R}_{\mathsf{even}}$ is weakly left-linear [5]. Therefore, by Theorem 9, $\mathcal{R}_{\mathsf{even}}$ is confluent.

To show the usefulness of our approach, we want to repeat a result of [13] using Theorem 9:

Corollary 11. *Orthogonal properly oriented right-stable 3-CTRSs are confluent.*

Proof. Orthogonal properly oriented right-stable 3-CTRSs are unraveled into orthogonal and therefore confluent TRSs by $\mathbb{U}_{conf}$. Therefore, we can apply Theorem 9. ☐

5 Conclusion and Perspectives

We have shown that unravelings can be used to prove confluence of CTRSs. In order to do this, we use soundness for joinability and a new unraveling, similar to the unraveling of [10], but with better properties concerning confluence while retaining soundness properties.

In the future, we want to show soundness for joinability of other classes of CTRSs and also use other transformations to analyze soundness properties.

A way to show joinability is the use of tree automata techniques developed to analyze reachability. The techniques are well investigated for TRSs, and they are very useful. However, the direct application of the techniques to CTRSs is very complicated and the constructed tree automata are often overapproximations (cf. [3]). Thus, unravelings would be very useful to analyze reachability and then confluence of CTRSs for which unravelings are sound. For this reason, we will also work for soundness of unravelings, e.g., to find soundness conditions.

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